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Finite-size scaling in arbitrary dimensions

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Abstract. We present here the results of an analytical study of certain thermodynamic properties of an ideal relativistic Bose gas confined to the curved space $\mathbb{S}^3 \times \mathbb{R}^{d-3}$, with $3 \leq d \leq 4$. As d approaches the marginal dimensionality 4, the amplitudes pertaining to these properties display a singularity similar to that displayed by the correlation length ξ_c of a spherical model of ferromagnetism in the flat-space geometry $L^{d-d'} \times \infty^{d'}$, with $2 < d \leq 4$ and $d' < 2$. These results are in sharp contrast with Cardy's generalization of ξ_c for a system in the space $\mathbb{S}^{d-1} \times \mathbb{R}^1$, with $2 < d < 4$.

1. Introduction

A few years ago Cardy (1985) gave a generalization of a well known result of conformal invariance, which is normally valid in two dimensions, to dimensionality greater than two. This result stems from the study of an infinitely long strip of width L subject to periodic boundary conditions and provides a direct connection between the critical-point correlation length ξ_c and the system size L, namely

$$\xi_c = L/2\pi x \tag{1}$$

where x is the scaling dimension of the corresponding scaling operator; for $d = 2$, $x = \frac{1}{2}\eta$. The generalization in question was carried out by effecting a conformal transformation of the metric of an infinite d-dimensional flat space $\mathbb{R}^d$ onto the metric of the space $\mathbb{S}^{d-1} \times \mathbb{R}^1$ which consists of an infinite one-dimensional flat space, $\mathbb{R}^1$, associated with a $(d-1)$-dimensional curved space, $\mathbb{S}^{d-1}$, of uniform radius R. Cardy inferred that relation (1) continued to hold for this case as well, with $L = 2\pi R$ and

$$x = \tfrac{1}{2}(d - 2 + \eta) \qquad 2 < d < 4. \tag{2}$$

An explicit calculation for the spherical model of ferromagnetism ($\eta = 0$) embedded in the space $\mathbb{S}^2 \times \mathbb{R}^1$ seemed to verify this result for the special case $d = 3$. We wonder if this would be true for *arbitrary* values of d—especially those close to 4.

In this connection we recall the corresponding results for a spherical model in the flat-space geometry $L^{d-d'} \times \infty^{d'}$ ($2 < d \leq 4$, $d' < 2$), namely

$$\xi_c \sim L \times \begin{cases} (4-d)^{-1/(4-d')} & d \leq 4 \quad (3a) \\ (\ln L)^{1/(4-d')} & d = 4 \quad (3b) \end{cases}$$

(see Brézin 1982, Luck 1985, Singh and Pathria 1986, 1987a), which show a very specific behaviour as the total dimensionality d of the system approaches the critical value 4; this behaviour is characterized by a singularity at $d = 4$ which, for a finite system,

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translates into an additional dependence on L through a logarithmic factor. One might argue that, for large L on one hand and large R on the other, the *qualitative* features of ξ_c would be the same—irrespective of the curvature of the space involved; the numerical factors, of course, will be different in the two situations. As it stands, relation (1), with x given by (2), does not possess the kind of feature displayed in expression (3a).

To examine this question further one would like to investigate the problem of a spherical-model system in curved geometries by carrying out the summations-over-states appearing in the various quantities pertaining to the system more accurately than is ordinarily done. In most cases, however, the eigenvalue spectrum of the problem and the associated multiplicities of the states make such an analysis formidable; this is indeed true of the geometry, $\mathbb{S}^{d-1}\times\mathbb{R}^1$, considered by Cardy. We, therefore, decided to consider an alternative space, $\mathbb{S}^3\times\mathbb{R}^{d'}$ ($d'\leqslant 1$), analysis for which can be carried out *exactly*. Even so, there are problems 'approximating the continuum of the curved space by a sequence of regular lattices': we avoid these by switching over to an ideal Bose gas, instead of the spherical model, for which no lattice structure is required. In view of the fact that these two systems belong to the same universality class, the results obtained for the Bose gas are expected to elucidate the problem of the spherical model as well. The final difficulty here lies in coining a proper definition of the correlation length ξ—one that holds at all temperatures T, especially at $T=T_c$. Our previous experience has shown (see, for instance, Singh *et al* 1986) that whenever the lowest eigenvalue of the spectrum is positive definite the limiting value of the parameter μ^2 (in Cardy's notation), as $T\to 0$, turns out to be negative; this not only creates problems of analytic continuation of various functions through the point where $\mu^2=0$ (which occurs at a *finite* temperature $T_0 \simeq T_c$) but also complicates the relationship between ξ and μ; see also Henkel (1988). This difficulty does not arise in Cardy's geometry but it does in ours. We, therefore, decided to examine quantities such as the (singular part of the) specific heat at constant volume, $c_\rho^{(s)}$, and the isothermal compressiblility, κ_T, of the system—rather than its correlation length. Of course, the concern relating to the limit $d\to 4_-$ is shared by all physical quantities that are singular in the bulk system at $T=T_c$.

In section 2 we set up analytical expressions for the particle density ρ and the energy density u of an ideal relativistic Bose gas, taking into account the possibility of particle-antiparticle pair production in the system; the desired quantities $c_\rho^{(s)}(R)$ and $\kappa_T(R)$ are then readily evaluated. In section 3 we examine these quantities in different regimes of T, especially at $T=T_c$, where we study their behaviour as a function of the total dimensionality d—in particular, for $d\leqslant 4$ and $d=4$. Remarkably enough, we encounter features similar to those displayed by the flat-space expressions (3a) and (3b); we suspect that the same would be true of the correlation length $\xi_c(R)$ as well. In view of the fact that, in the systems under consideration, mean-field behaviour takes over as soon as d exceeds 4, one would expect the singularity (in the finite-size amplitudes) at $d=4$ to be present, irrespective of the geometrical nature of the space available. If so, the generalization (2), though meant for a somewhat different geometry, may not hold for *all* values of d between 2 and 4.

2. Thermodynamics of an ideal relativistic Bose gas in geometry $\mathbb{S}^3\times\mathbb{R}^{d'}$

We consider an ideal Bose gas composed of N_1 particles and N_2 antiparticles, each of mass m, confined to the space $\mathbb{S}^3\times\mathbb{R}^{d'}$, with $d'\leqslant 1$. Since particles and antiparticles are supposed to be created in pairs, the system is governed by the conservation of the number Q $(=N_1-N_2)$, rather than of N_1 and N_2 separately; the conserved quantity Q may be looked upon as a kind of generalized 'charge'. In equilibrium the chemical potentials of the two species will be equal and opposite: $\mu_1=-\mu_2=\mu$, say, with the result that (Haber and Weldon 1981, 1982)

$$N_1=\sum_{\varepsilon}\{\exp[\beta(\varepsilon-\mu)]-1\}^{-1} \qquad N_2=\sum_{\varepsilon}\{\exp[\beta(\varepsilon+\mu)]-1\}^{-1} \tag{4}$$

where $\varepsilon=(m^2+k^2)^{1/2}$; for economy we employ units such that $\hbar=c=k_{\mathrm{B}}=1$. Note that both ε and μ include the rest energy m of the particle (or the antiparticle) and, for the mean occupation numbers in the various states to be positive definite, we must have $|\mu|\leqslant\varepsilon_{\min}$. Assuming that, to begin with, $\mu>0$, it readily follows that $N_1>N_2$ and hence $Q>0$. In view of the conservation of Q, μ then stays positive under all circumstances. Without loss of generality, we shall assume that this indeed is the case.

The eigenvalues, k_n, of the wavenumber k for a free particle confined to the space $\mathbb{S}^3$, which is well known as the Einstein universe, are given by (see, for example, Schrödinger 1938)

$$k_n=n/R \qquad n=1,2,3,\ldots \tag{5}$$

with multiplicity $g_n=n^2$. The particle energy ε in the space $\mathbb{S}^3\times\mathbb{R}^{d'}$ may, therefore, be written as

$$\varepsilon=(m^2+k_\parallel^2+n^2/R^2)^{1/2} \tag{6}$$

where $\boldsymbol{k}_\parallel$ is the wavevector associated with the flat space $\mathbb{R}^{d'}$, which may, henceforth, be treated as a continuous variable. The 'charge density' ρ is then given by

$$\rho\equiv\frac{N_1-N_2}{V}=\frac{2}{V}\sum_{j=1}^{\infty}\sinh(j\beta\mu)\int_{-\infty}^{\infty}\left(\frac{L_\parallel}{2\pi}\right)^{d'}d^{d'}k_\parallel\sum_{n=1}^{\infty}n^2$$
$$\times\exp\left[-j\beta m\left(1+\frac{k_\parallel^2}{m^2}+\frac{n^2}{m^2R^2}\right)^{1/2}\right] \tag{7}$$

where $V(=2\pi^2R^3L_\parallel^{d'})$ is the volume of the given space, with R finite and $L_\parallel$ going to infinity. Following the procedure laid down in our previous work on Bose-Einstein condensation in an Einstein universe (Singh and Pathria 1984, 1987b), we obtain after considerable algebra

$$\rho=\rho_{\mathrm{B}}(\beta,\mu)-\frac{\mu(m^2-\mu^2)^{(d-2)/2}}{2^{d-3}\pi^{d/2}\beta}\left[\mathscr{K}\left(\frac{d-4}{2}\middle|1;y\right)+\frac{d-3}{2}\mathscr{K}\left(\frac{d-2}{2}\middle|1;y\right)\right] \tag{8}$$

where $\rho_{\mathrm{B}}(\beta,\mu)$ is the corresponding bulk result:

$$\rho_{\mathrm{B}}(\beta,\mu)=\frac{m^d}{2^{(d-3)/2}\pi^{(d+1)/2}}\sum_{j=1}^{\infty}\frac{\sinh(j\beta\mu)}{(j\beta m)^{(d-1)/2}}K_{(d+1)/2}(j\beta m) \tag{9}$$

$K_\nu(z)$ being the modified Bessel functions, while the remaining terms represent finite-size effects in the system; here,

$$\mathscr{K}(\nu|l;y)=\sum_{q(l)}{}'\frac{K_\nu(2yq)}{(yq)^\nu} \qquad q=(q_1^2+\ldots+q_l^2)^{1/2}>0 \tag{10}$$

so that

$$\mathscr{K}(\nu|1;y)=\sum_{q=-\infty}^{\infty}{}'\frac{K_\nu(2y|q|)}{(y|q|)^\nu}=2\sum_{q=1}^{\infty}\frac{K_\nu(2yq)}{(yq)^\nu} \tag{11}$$

while y is the thermogeometric parameter of the problem which scales R with μ:

$$y = \pi(m^2 - \mu^2)^{1/2} R. \tag{12}$$

The bulk critical temperature, β_c, is determined by the (obvious) condition

$$\rho_B(\beta_c, m) = \rho. \tag{13}$$

In the region of phase transition ($\mu \simeq m$), the bulk term in (8) takes the form

$$\rho_B(\beta, \mu) = \rho_B(\beta, m) - \frac{m}{2^{d-1}\pi^{d/2}\beta}\left|\Gamma\left(\frac{2-d}{2}\right)\right|(m^2-\mu^2)^{(d-2)/2} + O(m^2-\mu^2)^1 \tag{14a}$$

if $d < 4$, whereas

$$\rho_B(\beta, \mu) = \rho_B(\beta, m) - \frac{m}{8\pi^2\beta}(m^2-\mu^2)\left[\ln\left(\frac{m}{\beta(m^2-\mu^2)}\right) + \text{constant}\right] + O(m^2-\mu^2)^2 \tag{14b}$$

if $d = 4$; the undetermined constant appearing in (14b) is of order unity. At the same time, the sums appearing in the finite-size terms of (8) can be simplified by using identities from Singh and Pathria (1989), whereby

$$\begin{aligned}\mathcal{K}\left(\frac{d-4}{2}\middle|1; y\right) &+ \frac{d-3}{2}\mathcal{K}\left(\frac{d-2}{2}\middle|1; y\right)\\ &= \frac{d-3}{2}\Gamma\left(\frac{d-2}{2}\right)\zeta(d-2)\frac{1}{y^{d-2}} + \tfrac{1}{4}\Gamma\left(\frac{2-d}{2}\right)\\ &\quad + \pi^{1/2}\Gamma\left(\frac{5-d}{2}\right)\frac{1}{y^{d-2}}\sum_{q=1}^{\infty}\left((\pi q)^{d-3} - \frac{\pi^2 q^2}{(\pi^2 q^2 + y^2)^{(5-d)/2}}\right)\end{aligned} \tag{15a}$$

if $d < 4$, whereas

$$\begin{aligned}\mathcal{K}(0|1; y) &+ \tfrac{1}{2}\mathcal{K}(1|1; y)\\ &= \frac{\pi^2}{12y^2} + \tfrac{1}{2}\ln\left(\frac{y}{2\pi}\right) + \tfrac{1}{2}\gamma + \tfrac{1}{4} + \frac{\pi}{y^2}\sum_{q=1}^{\infty}\left(\pi q - \frac{y^2}{2\pi q} - \frac{\pi^2 q^2}{(\pi^2 q^2 + y^2)^{1/2}}\right)\end{aligned} \tag{15b}$$

if $d = 4$; here, γ is the well known Euler constant. Substituting (14) and (15) into (8), we obtain to the desired order in y/R

$$\begin{aligned}\rho = \rho_B(\beta, m) &- \frac{m}{2^{d-3}\pi^{(3d-4)/2}\beta R^{d-2}}\\ &\times\left[\frac{d-3}{2}\Gamma\left(\frac{d-2}{2}\right)\zeta(d-2)\right.\\ &\left. + \pi^{1/2}\Gamma\left(\frac{5-d}{2}\right)\sum_{q=1}^{\infty}\left((\pi q)^{d-3} - \frac{\pi^2 q^2}{(\pi^2 q^2 + y^2)^{(5-d)/2}}\right)\right]\end{aligned} \tag{16a}$$

if $d < 4$, whereas

$$\begin{aligned}\rho = \rho_B(\beta, m) &- \frac{m}{2\pi^4\beta R^2}\\ &\times\left\{\frac{\pi^2}{12} + y^2\left[\tfrac{1}{2}\ln\left(\frac{R}{\Lambda}\right) + \text{constant}\right]\right.\\ &\left. + \pi\sum_{q=1}^{\infty}\left(\pi q - \frac{y^2}{2\pi q} - \frac{\pi^2 q^2}{(\pi^2 q^2 + y^2)^{1/2}}\right)\right\}\end{aligned} \tag{16b}$$

if $d=4$; here, Λ ($=\sqrt{2\pi\beta/m}$) denotes the mean thermal wavelength of the particles in the system. For a given ρ, equations (16) determine y as a function of β and R.

For the study of the specific heat, we evaluate the energy density u of the system, which is given by

$$u=\frac{2m}{V}\sum_{j=1}^{\infty}\cosh(j\beta\mu)\int_{-\infty}^{\infty}\left(\frac{L_{\|}}{2\pi}\right)^{d'}d^{d'}k_{\|}\sum_{n=1}^{\infty}n^2\left(1+\frac{k_{\|}^2}{m^2}+\frac{n^2}{m^2R^2}\right)^{1/2}$$
$$\times\exp\left[-j\beta m\left(1+\frac{k_{\|}^2}{m^2}+\frac{n^2}{m^2R^2}\right)^{1/2}\right]. \tag{17}$$

Analysing (17) in the same manner as (7), we obtain for the singular part of the specific heat per unit volume

$$c_\rho^{(s)}=\frac{\beta^2 y}{\pi^2 mR^2}\left(\rho-\rho_B(\beta,m)-\beta\frac{\partial\rho_B(\beta,m)}{\partial\beta}\right)\left(\frac{\partial y}{\partial\beta}\right)_\rho. \tag{18}$$

The derivative $(\partial y/\partial\beta)_\rho$ can be obtained from equations (16), leading to the final result

$$c_\rho^{(s)}=-\frac{\pi^{3(d-3)/2}\beta^2(\rho-\rho_B(\beta,m)-\beta\partial\rho_B(\beta,m)/\partial\beta)^2}{2^{4-d}\Gamma\{(7-d)/2\}m^2R^{4-d}\sum_{q=1}^{\infty}[\pi^2q^2/(\pi^2q^2+y^2)^{(7-d)/2}]} \tag{19a}$$

for $d<4$, and

$$c_\rho^{(s)}=-\frac{2\pi^2\beta^2(\rho-\rho_B(\beta,m)-\beta\,\partial\rho_B(\beta,m)/\partial\beta)^2}{m^2\{\ln(R/\Lambda)+\text{constant}-\pi\sum_{q=1}^{\infty}[(1/\pi q)-\pi^2q^2/(\pi^2q^2+y^2)^{3/2}]\}} \tag{19b}$$

for $d=4$.

The isothermal compressibility of the sytem can be obtained straightforwardly with the help of the general relationship (see Singh and Pathria 1987b, c)

$$c_\rho^{(s)}\kappa_T=-\beta(\rho-\rho_B(\beta,m)-\beta\,\partial\rho_B(\beta,m)/\partial\beta)^2/\rho^2 \tag{20}$$

which does not depend on R; in view of this relationship, we may in the following concentrate on only one of these quantities, say $c_\rho^{(s)}$, and refer to the other only when it is deemed necessary. For determining the precise manner in which the quantities $c_\rho^{(s)}$ and κ_T vary with the parameters, β and R, of the problem, we have to eliminate y with the help of equations (16).

3. Thermodynamic behaviour in different regimes of temperature

Regime (i). In the region of *first-order* phase transition ($T<T_c$), the chemical potential μ of the system tends to the lowest eigenvalue of the particle energy ε, i.e. $\mu\to\varepsilon_{\min}=(m^2+1/R^2)^{1/2}\simeq m+1/(2mR^2)$, with the result that $y^2=\pi^2(m^2-\mu^2)\,R^2\to-\pi^2$. The slight but significant variation of y in this regime is given by (see equations (16), remembering that, for $\beta>\beta_c$, $\rho_B(\beta,m)<\rho_B(\beta_c,m)=\rho$)

$$(y^2+\pi^2)\approx\left(\frac{\Gamma\{(5-d)/2\}m}{2^{d-3}\pi^{3(d-3)/2}\beta\{\rho-\rho_B(\beta,m)\}R^{d-2}}\right)^{2/(5-d)} \tag{21}$$

for $d\leqslant 4$. Equations (19) then give

$$c_\rho^{(s)}\approx-\frac{\beta(\rho-\rho_B(\beta,m)-\beta\,\partial\rho_B(\beta,m)/\partial\beta)^2}{(5-d)\pi^2m\{\rho-\rho_B(\beta,m)\}R^2}$$
$$\times\left(\frac{\Gamma\{(5-d)/2\}m}{2^{d-3}\pi^{3(d-3)/2}\beta\{\rho-\rho_B(\beta,m)\}R^{d-2}}\right)^{2/(5-d)} \tag{22}$$

again for $d \leq 4$. As $T \to 0$, $\rho_B(\beta, m) \approx (m/2\pi\beta)^{d/2}\zeta(d/2) \to 0$ (see equation (9)), with the result that

$$c_\rho^{(s)}|_{T\to 0} \approx -\frac{1}{(5-d)\pi^2}\left(\frac{\Gamma\{(5-d)/2\}}{2^{d-3}\pi^{3(d-3)/2}}\right)^{2/(5-d)}\left(\frac{m}{\rho\beta}\right)^{(d-3)/(5-d)}\left(\frac{1}{R}\right)^{6/(5-d)}. \tag{23}$$

It seems worthwhile to point out here that the dependence of $c_\rho^{(s)}$ on R for all $T < T_c$ *and* on T as $T \to 0$ are consistent with the flat-space results reported earlier (see Singh and Pathria 1987c).

Regime (ii). For $T \geq T_c$ and $R \to \infty$, we recover standard bulk results, with finite-size corrections that are exponentially small. For the record, we note that in this regime

$$y \sim Rt^{1/(d-2)} \gg 1 \qquad t = \frac{T - T_c}{T_c} \approx \frac{\beta_c - \beta}{\beta_c} \quad 0 < t \ll 1 \tag{24}$$

with the result that

$$|c_\rho^{(s)}| \sim \begin{cases} \dfrac{1}{R^{4-d}y^{d-4}} \sim t^{(4-d)/(d-2)} & d < 4 \qquad (25a) \\[2ex] \dfrac{1}{\ln(R/\Lambda_c) - \ln y} \sim \dfrac{1}{\ln(1/t)} & d = 4. \qquad (25b) \end{cases}$$

Regime (iii). Between regimes (i) and (ii) lies the 'core' region where both y^2 and $(y^2 + \pi^2)$ are of order unity. In this region we identify two special points—$T = T_0$ where $y = 0$ and $T = T_c$ where $y = y_c$. Equations (16) tell us that $T_0 > T_c$ and that $(T_0 - T_c) \sim 1/R^{d-2}$. Since y^2 is a monotonically increasing function of T, it follows that y_c^2 is negative; its precise value is given by the implicit equation

$$\sum_{q=1}^{\infty}\left((\pi q)^{d-3} - \frac{\pi^2 q^2}{(\pi^2 q^2 + y_c^2)^{(5-d)/2}}\right) = -\frac{d-3}{2\pi^{1/2}}\frac{\Gamma\{(d-2)/2\}}{\Gamma\{(5-d)/2\}}\zeta(d-2) \tag{26a}$$

for $d < 4$, and by

$$y_c^2\left[\tfrac{1}{2}\ln\left(\frac{R}{\Lambda_c}\right) + \text{constant}\right] + \pi\sum_{q=1}^{\infty}\left(\pi q - \frac{y_c^2}{2\pi q} - \frac{\pi^2 q^2}{(\pi^2 q^2 + y_c^2)^{1/2}}\right) = -\frac{\pi^2}{12} \tag{26b}$$

for $d = 4$. For $d = 3$, we obtain the exact result: $y_c^2 = -\pi^2/4$; for other values of d, y_c^2 has to be determined numerically. For $d \leq 4$, we find that

$$y_c^2 \approx -\frac{\pi^2}{6}(4-d) \tag{27a}$$

while for $d = 4$

$$y_c^2 \approx -\frac{\pi^2}{6[\ln(R/\Lambda_c) + \text{constant}]}. \tag{27b}$$

It follows that

$$c_\rho^{(s)}|_{T=T_c} \approx -2\pi^2\left(\frac{\beta^2}{m}\frac{\partial\rho_B(\beta, m)}{\partial\beta}\right)_c^2 \times \begin{cases} \varepsilon/R^\varepsilon & \varepsilon = (4-d) \ll 1 \qquad (28a) \\ 1/[\ln(R/\Lambda_c) + \text{constant}] & d = 4. \qquad (28b) \end{cases}$$

In view of (20), the corresponding expressions for the isothermal compressibility turn out to be

$$\kappa_T|_{T=T_c} \approx \frac{m^2}{2\pi^2\beta_c\rho^2} \times \begin{cases} R^\varepsilon/\varepsilon & \varepsilon=(4-d)\ll 1 \quad (29a) \\ \ln(R/\Lambda_c)+\text{constant} & d=4. \quad (29b) \end{cases}$$

4. Discussion of results

First of all we observe that the specific heat $c_\rho^{(s)}$ (and, by implication, the isothermal compressiblity κ_T) of the system under study possess an R dependence such that

(i) for $T<T_c$, it is strictly algebraic—with exponent $2(d-d')/(2-d')$ $(=6/(5-d)$, for $d=3+d')$, which is formally the same for all $d\leqslant 4$, while

(ii) for $T=T_c$, it is algebraic for $d<4$—with exponent $(4-d)$ and an amplitude that tends to vanish (or diverge) as $d\to 4_-$, paving the way for the R dependence to become logarithmic at $d=4$; for $T\gtrsim T_c$, we encounter a similar dependence on t. Secondly, and more importantly, we observe that, for $d\leqslant 4$ and for $d=4$, the R dependence of the quantities $c_\rho^{(s)}|_{T=T_c}$ and $\kappa_T|_{T=T_c}$ in the space $\mathbb{S}^3\times\mathbb{R}^{d'}$ is very much similar to the L dependence of the quantity ξ_c in the flat-space geometry $L^{d-d'}\times\infty^{d'}$; compare equations (28) and (29) with (3). It is conceivable that ξ_c in the space $\mathbb{S}^3\times\mathbb{R}^{d'}$ (or, for that matter, in any space, including $\mathbb{S}^{d-1}\times\mathbb{R}^1$) would also possess similar features as $d\to 4_-$.

Ideally we ourselves should have evaluated ξ_c in the space $\mathbb{S}^3\times\mathbb{R}^{d'}$. We could not do so because we do not have at our disposal a relationship between ξ and y that holds at *all* temperatures T. While for $T\gtrsim T_c$, where $y\gg 1$, ξ is known to be $\sim R/y\sim t^{-1/(d-2)}$ and for $T<T_c$, where $y^2\simeq-\pi^2$, ξ is presumably $\sim R/(y^2+\pi^2)^{1/2}\sim R^{3/(5-d)}$, we have no clear idea how ξ varies with y in the 'core' region where y^2 passes from positive to negative values. This difficulty arises in all cases where the lowest eigenvalue of the energy spectrum is such that $\varepsilon_{\min}>m$, with the result that, as the temperature of the system is lowered, the chemical potential μ eventually approaches this limiting value and in the process makes y^2 negative; see Singh *et al* (1986) and Henkel (1988). In any case, whatever the relationship between ξ and y, the quantity ξ_c, for $d\to 4_-$, should in our opinion possess the kind of feature emphasized here.

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